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COMPLEX INTEGRATION WITH PARTIAL-FRACTION DECOMPOSITION TECHNIQUE AND GENERALIZING METHOD FOR CERTAIN COMPLEX FUNCTIONS

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ABSTRACT

We know that calculating complex integrals is not as easy as calculating real integrals. Also, the complex integration techniques in comparison to real integration techniques are limited. In this paper we attempt to introduce partial-fraction decomposition technique of real integration for complex integration and with respect to the nature of complex integrals, generalize this method for a certain type of complex integral function that has no standard conditions.

Keywords: Real function, complex function, rational complex function, real integration, complex integration, partial-fraction decomposition.

INTRODUCTION

To calculate real integral $\int f(x)dx$, if the function under the integral $f(x)$ is a rational function in the form $\frac{p_m(x)}{p_n(x)}$, one of the best methods to integrate is partial-fraction decomposition technique (Maron, 1970) .On the other hand, in the general case, we know that calculation of complex integral typically, is done by one of the following methods: 1) Using complex integration definition or primitive function, 2) Using Cauchy or Cauchy-Goursat theorem, and their generalization, 3) Applying Cauchy's integral formula and its generalization, and 4) Using the residue theorem and Laurent expansions. Each of aforementioned methods has their own terms and difficulty in calculating (Brown, and Churchill, 1996). In the present study, we want to combine partial-fraction decomposition integration with methods 2 or 3, and introduce partial-fraction decomposition integration method for complex integration, and then generalize it for some certain complex functions which have no standard conditions to use it. First we review methods 2 and 3 as it is required in this paper.

Cauchy Theorem: Let function f be analytic at each point inside and on simple closed contour c and f' be continuous at each point inside c . Then we have

$$\oint_c f(z)\,dz = 0$$

Proof. Assume that parametric equation of simply-closed c is as below which is in positive direction:

$$a \le t \le b$$

$$Z(t) = x(t) + iy(t)$$

And let function $f(z) = u(x, y) + iv(x, \mathrm{y})$ be inside and on simple closed contour c. Then according to complex integral definition, we have

$$\oint_c f(z)\,dz = \int_b^a f(z(t)).z'(t)\,dt$$

Also, we know

$$f(z(t)).z'(t) = \left(u(t)x'(t) - v(t)y'(t)\right) + i\left(v(t)x'(t) + u(t)y'(t)\right)$$

Therefore,

$$\oint_c f(t)\,dt = \int_b^a (ux' - vy')\,dt + i\int_b^a (vx' + uy')\,dt$$

Since

$$dx = x'(t)\,dt, dy = y'(t)\,dt$$

Then

$$\oint_c f(z)\,dz = \int_c (udx - vdy) + i\int_c (vdx + udy)$$

Assume that R is at any point inside simply-closed contour c. since function f is continuous in R, then v and u are continuous, and since f' is continuous, then the first-order partial derivatives of v and u in R are

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continuous. Therefore, according to Green's theorem we have

$$\oint_c f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dA + i\iint_R \left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dA$$

But according to Cauchy-Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Hence both integrals in the right side of above equation are zero, thus $\oint f(z)dz = 0$

Cauchy-Goursat theorem: Let function *f* be analytic at each point inside and on simple closed contour *c*. Then we have $\oint_c f(z)\,dz = 0$.

Proof. Brown and Churchill (1996)

Example 1: find complex integral $\oint_c \frac{z^2}{z-3}dz$ where *c* is contour of positively oriented circle $|z| = 1$.

Since function $\frac{z^2}{z-3}$ is analytic at any point inside simply-closed contour *c*, then based on Cauchy-Goursat theorem, the resulting integral is zero.

Example 2: evaluate complex integral

$\oint_c \frac{1}{z^2+2z+2}dz = 0$ where *c* is contour of positively oriented circle $|z| = 1$.

Since function $\frac{1}{z^2+2z+2}$ is analytic at any point inside simply-closed contour *c*, then based on Cauchy-Goursat theorem, the resulting integral is zero.

Note: Cauchy-Goursat theorem for multiple connected domains is also established.

Example 3: find complex integral

$\oint_{\partial D} \frac{z+2}{\sin\left(\frac{z}{2}\right)}dz$ where ∂D is the contour of domain between circle |z|=4 and a rectangle with sides along x = ± 1 and y = ± 1 in positive direction.

Since function $\frac{z+2}{\sin\left(\frac{z}{2}\right)}$ is analytic at any point inside and on multiply- connected domain *D,* then

$$\oint_{\partial D} \frac{z+2}{\sin\left(\frac{z}{2}\right)}dz = 0$$

Example 4: evaluate complex integral $\oint_{\partial D} \frac{1}{3z^2+1}dz$ where ∂D is the contour of domain between circle |z|=4 and a rectangle with sides along x = ± 1 and y = ± 1 in positive direction.

Since function $\frac{1}{3z^2+1}$ is analytic at any point inside and on multiply- connected domain *D,* then $\oint_{\partial D} \frac{1}{3z^2+1}dz = 0$.

Cauchy's integral formula: suppose *f* is analytic at any point inside and on simply-closed contour *c* in positive direction and z_0 is a point inside *c*. then:

$$f(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-z_0}dz$$

Proof. since *f* is continuous in z_0, hence
$\forall \varepsilon > 0, \exists\, \sigma > 0;\ \forall z(|z - z\_0| < \sigma \Rightarrow |f(z) - f(z\_0)| < \varepsilon$

Now let an arbitrary positive number ρ be less than σ and circle $C_0\,|z - z_0| = \rho$ be inside or on simply-closed contour *c* in positive direction. Then
$|z - z_0| = \rho \Rightarrow |f(z) - f(z_0)| < \varepsilon$

Since function $\frac{f(z)}{z - z_0}$ is analytic in a closed contour including paths *c* and C_0, and at all points between them. According to Brown and Churchill (1996), we know

$$\oint_c \frac{f(z)}{z-z_0}dz = \oint_{c_0} \frac{f(z)}{z-z_0}dz$$

So we can write

$$\oint_c \frac{f(z)}{z-z_0}dz - f(z_0)\oint_{c_0} \frac{1}{z-z_0}dz = \oint_c \frac{f(z)-f(z_0)}{z-z_0}dz$$

On the hand, we know

$$\oint_{c_0} \frac{1}{z-z_0}dz = 2\pi i$$

Then

$$\oint_c \frac{f(z)}{z-z_0}dz - 2\pi i\, f(z_0) = \oint_c \frac{f(z)-f(z_0)}{z-z_0}dz$$

Since the length of C_0 is equal to $2\pi\rho$, based on Brown and Churchill (1996), we can write

$$\left|\int_c \frac{f(z)-f(z_0)}{z-z_0}dz\right| < \frac{E}{\rho}2\pi\rho = 2\pi\varepsilon$$

Therefore,

$$\left|\oint_c \frac{f(z)}{z-z_0}dz - 2\pi i\, f(z_0)\right| < 2\pi\varepsilon$$

Since the right-hand side of this inequality is non-negative integer constant and is smaller than any small arbitrary positive number, it should be zero. This completes the proof.

Generalization of Cauchy's integral formula: let function f be analytic at all points inside and on simply closed contour c , and z_0 be a point on c. then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Proof. First we show that

$$f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}dz$$

Let $0 < |\Delta z| < d$, where d is the shortest distance of z_0 to points Z on c. then, following Cauchy's integral formula, we have

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_c\left(\frac{1}{z-z_0-\Delta z}-\frac{1}{z-z_0}\right)\frac{f(z)}{\Delta z}dz = \frac{1}{2\pi i}\oint_c \frac{1}{(z-z_0-\Delta z)(z-z_0)}dz$$

Now, since f is continuous on c, then we want to show that, letting $\Delta z \to 0$, The latter integral tends to $\oint_c \frac{f(z)}{(z-z_0)}dz$. To do this, first we write the difference between the two integrals

$$\oint_c\left[\frac{1}{(z-z_0-\Delta z)(z-z_0)} - \frac{1}{(z-z_0)^2}\right]f(z)dz$$

as

$$\Delta z\oint_c \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)^2}dz$$

Then we assume M denotes maximum value of $|f(z)|$ on c and L is the length of c. Note that $|z-z_0| \geq d$, and $|z-z_0-\Delta z| \geq \left||z-z_0|-|\Delta z|\right| \geq d-|\Delta z|$. Thus

$$\left|\Delta z\oint_c \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)^2}dz\right| \leq \frac{|\Delta z|ML}{(d-|\Delta z|)d^2}$$

where the last fraction tends to be zero. If $\Delta z \to 0$, thus

$$\lim_{\Delta z\to 0}\oint_c \frac{f(z+\Delta z)-f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^2}dz$$

Then

$$f'(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^2}dz$$

Similarly, the theorem for the nth order derivative is also proved.

Note: Cauchy's integral formula for multiple connected domains is also established if f is analytic at any point inside and on multiply-connected domain D in positive direction and z_0 is a point inside domain D. then:

$$f(z_0) = \frac{1}{2\pi i}\oint_{\partial D}\frac{f(z)}{(z-z_0)}dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_{\partial D}\frac{f(z)}{(z-z_0)^{n+1}}dz$$

Using Cauchy's integral formula and its generalization to calculate complex integrals (method 3): if function $f(z)$ is analytic on and inside multiply-connected domain D or simply- closed contour c *in* positive direction, and let z_0 be a point inside D or c. then

$$\oint_{\partial D \text{ or } c}\frac{f(z)}{(z-z_0)}dz = 2\pi i\, f(z_0)$$

$$\oint_{\partial D \text{ or } c}\frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z_0)$$

Example 5: find complex integral $\oint_c \frac{z}{(9-z^2)(z+1)}$ where c is contour of positively oriented circle $|z| = 2$.

Let function $\frac{z}{9-z^2}$ be analytic throughout c, and $-i$ be a point inside circle. According to Cauchy's integral formula we can write

$$\oint_c \frac{z}{(9-z^2)(z+i)}dz = 2\pi i\left(\frac{-i}{9-|-i|^2}\right) = \frac{\pi}{5}$$

Example 6: evaluate complex integral $\oint_c \frac{\exp(2z)}{z^4}dz$ where c is contour of positively oriented circle $|z| = 1$.

Let function $\exp(2z)$ be analytic on and inside c, and 0 be a point inside circle. According to Cauchy's integral formula we have

$$\oint_c \frac{\exp\exp(2z)}{z^4}dz = \frac{2\pi i}{3!}\left((\exp(2z))'''\right|z=0 = \frac{2\pi i}{3}$$

Example 7: evaluate complex integral $\oint_c \frac{1}{\left(z^2+4\right)^2} dz$ where *c is* $|z-i| = 2$.

Let function $\frac{1}{(z+2i)^2}$ be analytic on and inside *c*, and $2i$ be a point inside circle. We have

$$\oint_c \frac{1}{\left(z^2+4\right)^2} dz = \oint_c \frac{1}{(z+2i)^2(z-2i)^2} dz = \frac{2\pi i}{1!}\left(\left(\frac{1}{(z+2i)^2}\right)'\right|z=2i = \frac{\pi}{16}$$

Example 8: find complex integral $\oint_c \frac{1}{z^2+4} \mathrm{dz}$ where *c is* $|z-i| = 2$.

Let function $\frac{1}{z^2+4}$ be analytic on and inside *c*, and $2i$ be a point inside circle. Thus

$$\oint_c \frac{1}{z^2+4} \mathrm{dz} = 2\pi i\left(\frac{1}{2i+2i}\right) = \frac{\pi}{2}$$

Example 9: find $\oint_{\partial D} \frac{\cos(z-1)}{(z-1)(z^2+9)} dz$ where ∂D is the contour of multiply- connected domain $\frac{1}{2} < |z| < 2$ in positive direction.

$$\oint_{\partial D} \frac{\cos(z-1)}{(z-1)(z^2+9)} dz = 2\pi i\left(\frac{\cos(1-1)}{1+9}\right) = i\frac{\pi}{5}$$

Example 10: find $\oint_{\partial D} \frac{z+1}{\left(z^2-4\right)^2} \mathrm{dz}$ where ∂D is the contour of multiply- connected domain $\frac{1}{2} < |z-1| < 2$ in positive direction.

We have:

$$\oint_{\partial D} \frac{z+1}{\left(z^2-4\right)^2} \mathrm{dz} = \frac{2\pi i}{1!}\left(\left(\frac{z+1}{(z+2)^2}\right)'\right|z=2 = -\frac{\pi i}{16}$$

Complex Integration with Partial-Fraction Decomposition Technique

According to the fundamental theorem of algebra, we know that every n^{th} degree polynomial $p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_\circ \qquad (i = 1, \ldots, n)$ $a_i \in \mathbb{C}$ has *n* complex root [2] .In general, the roots can be divided into two categories: a) Distinct roots as $z_1, z_2, \ldots z_n$, and b) Non-distinct roots and some of them as $z_{i_1}, \cdots, z_{i_k}$ which are repetition of $r_1, \cdots, r_k$ degree. In this case, according to (a), n^{th} degree polynomial $p_n(z)$, can be decomposed as:

$p_n(z) = a_n(z-z_1)(z-z_2)\ldots(z-z_n)$ and according to (b), n^{th} degree polynomial $p_n(z)$ can be decomposed as $p_n(z) = a_n(z-z_1)\ldots(z-z_{i_1})^{r_1}\ldots(z-z_{i_k})^{r_k}\ldots(z-z_N)$. [2]

Now we consider the complex integral $\oint_c f(z)dz$ where *c* is a positively-oriented simple closed contour and $f(z)$ is a complex rational function in the form $\frac{p_m(z)}{p_n(z)}$ ($m < n$). Assume that more than one of $p_n(z)$ roots is inside simple closed contour (otherwise integration will be calculated by one of the methods 2, 3, or 4), Then we introduce one of following integration modes as complex integration with partial-fraction decomposition technique:

- If the roots of n^{th} degree polynomial $p_n(z)$ is like type (a), the partial fraction decomposition of $\frac{p_m(z)}{p_n(z)}$ is given by

$$\frac{p_m(z)}{p_n(z)} = \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \ldots + \frac{A_n}{z-z_n} \qquad (1)$$

Where $A_1 \ldots A_n$ are distinct constants that are solved by denomination of the right-hand fraction and equating coefficients of terms in polynomials of two sides of fraction, thus

$$\oint_c f(z)dz = \oint_c \frac{A_1}{z-z_1}dz + \oint_c \frac{A_2}{z-z_2}dz + \ldots + \oint_c \frac{A_n}{z-z_n}dz \qquad (2)$$

Each of above integrals can be easily calculated by Cauchy or Cauchy-Goursat theorem.

- If the roots of n^{th} degree polynomial $p_n(z)$ is like type (b), The partial fraction decomposition of $\frac{p_m(z)}{p_n(z)}$ is given by

$$\frac{p_m(z)}{p_n(z)} = \frac{A_1}{z-z_1} + \ldots + \frac{B_{11}}{z-z_{i_1}} + \ldots + \frac{B_{1r_1}}{(z-z_{i_1})^{r_1}} + \ldots + \frac{B_{kr_k}}{(z-z_{i_k})^{r_k}} + \ldots + \frac{A_N}{z-z_N} \qquad (3)$$

Where A and B are distinct constants which are solved by denomination of the right-hand fraction and equating coefficients of terms in polynomials of two sides of fraction, thus

$$\oint_c f(z)dz = \oint_c \frac{A_1}{z-z_1}dz + \ldots + \oint_c \frac{B_{11}}{z-z_{i_1}}dz + \ldots + \oint_c \frac{B_{1r_1}}{(z-z_{i_1})^{r_1}}dz + \ldots + \oint_c \frac{B_{kr_k}}{(z-z_{i_k})^{r_k}}dz + \ldots + \oint_c \frac{A_N}{z-z_N}dz \qquad (4)$$

Each of above integrals also can be easily calculated by Cauchy or Cauchy-Goursat theorem.

Here are some examples:

Example 11: find complex integral $\oint_c \frac{3z^2-4}{z^2-3z+2}dz$ where c is contour of positively oriented circle $|z|=3$.

Since the degree of polynomials of the numerator and denominator are equal, first divide the denominator into the numerator:

$$\frac{3z^2-4}{z^2-3z+2}=3+\frac{9z-10}{z^2-3z+2}$$

In this case, the roots of right-hand polynomial z^2-3z+2 can be decomposed as:

$$\frac{3z^2-4}{z^2-3z+2}=3+\frac{1}{z-1}+\frac{8}{z-2}$$

Thus

$$\oint_c \frac{3z^2-4}{z^2-3z+2}dz=\oint_c 3dz+\oint_c \frac{1}{z-1}dz+\oint_c \frac{8}{z-2}dz$$
$$=\circ+2\pi i(1)+2\pi i(8)=18\pi i$$

Example 12: find complex integral $\oint_c \frac{z^3+4}{(z^4+2z^3-3z^2-8z-4)}dz$ where c is contour of positively oriented circle $|z|=4$.

The partial fraction decomposition is given by

$$\frac{z^3+4}{(z^4+2z^3-3z^2-8z-4)}=\frac{\frac{-1}{3}}{(z+1)}-\frac{1}{(z+1)^2}+\frac{\frac{1}{3}}{(z-2)}+\frac{1}{(z+2)}$$

Thus

$$\oint_c \frac{z^3+4}{(z^4+2z^3-3z^2-8z-4)}dz=\oint_c \frac{\frac{-1}{3}}{(z+1)}dz-\oint_c \frac{1}{(z+1)^2}dz+\oint_c \frac{\frac{1}{3}}{(z-2)}dz+\oint_c \frac{1}{z+2}dz$$
$$=2\pi i(\frac{-1}{3})-\circ+2\pi i(\frac{1}{3})+2\pi i(1)=2\pi i$$

Example 13: evaluate complex integral $\oint_c \frac{z^2-1}{z^4-2iz^3-z^2}dz$ where c is contour of positively oriented circle $|z|=2$.

The roots of polynomial of denominator can be decomposed as:

$$\frac{z^2-1}{z^4-2iz^3-z^2}=\frac{\frac{3i}{5}}{z}+\frac{1}{z^2}-\frac{\frac{3i}{10}}{(z-i)}+\frac{1}{(z-i)^2}$$

Thus

$$\oint_c \frac{z^2-1}{z^4-2iz^3-z^2}dz=\oint_c \frac{\frac{3i}{5}}{z}dz+\oint_c \frac{1}{z^2}dz-\oint_c \frac{\frac{3i}{10}}{(z-i)}dz+\oint_c \frac{1}{(z-i)^2}dz$$
$$=2\pi i(\frac{3i}{5})+\circ-2\pi i(\frac{3i}{10})+\circ=\frac{-3\pi}{5}$$

Generalization of the method for non-standard complex functions

If a complex function under the integral $\oint_c f(z)dz$ is not rational but as the form $\frac{g(z)}{p_n(z)}$ provided that the function $g(z)$ is analytic at each point inside and on simple closed contour c, integration with partial-fraction decomposition can be done as following:

- If the roots of n^{th} degree polynomial $p_n(z)$ is like type (a), then rational function $\frac{1}{p_n(z)}$ can be decomposed as:

$$\frac{1}{p_n(z)}=\frac{A_1}{z-z_1}+\frac{A_2}{z-z_2}+...+\frac{A_n}{z-z_n} \qquad (5)$$

Where A_1... A_n are distinct constants solved by denomination of the right-hand fraction and equating coefficients of terms in polynomials of two sides of fraction, thus

$$\oint_c f(z)dz=\oint_c \frac{A_1 g(z)}{z-z_1}dz+\oint_c \frac{A_2 g(z)}{z-z_2}dz+...+\oint_c \frac{A_n g(z)}{z-z_n}dz \qquad (6)$$

Again, each of above integrals can be easily calculated by Cauchy or Cauchy-Goursat theorem.

- If the roots of n^{th} degree polynomial $p_n(z)$ is like type (b), then rational function $\frac{1}{p_n(z)}$ can be decomposed as:

$$\frac{1}{p_n(z)}=\frac{A_1}{z-z_1}+...+\frac{B_{11}}{(z-z_{i_1})}+...+\frac{B_{1r_1}}{(z-z_{i_1})^{r_1}}+...+\frac{B_{k1}}{(z-z_{i_k})}+...+\frac{B_{kr_k}}{(z-z_{i_k})^{r_k}}+...+\frac{A_N}{z-z_N} \qquad (7)$$

Where A and B are distinct constants solved by denomination of the right-hand fraction and equating coefficients of terms in polynomials of two sides of fraction, thus

$$\oint_c f(z)dz=\oint_c \frac{A_1 g(z)}{z-z_1}dz+...+\oint_c \frac{B_{11} g(z)}{(z-z_{i_1})}dz+...+\oint_c \frac{B_{1r_1} g(z)}{(z-z_{i_1})^{r_1}}dz+...+\oint_c \frac{A_N}{z-z_N}dz \qquad (8)$$

Each of above integrals can be easily calculated by Cauchy or Cauchy-Goursat theorem.

Example 14: get $\oint_c \frac{2(1+\sin(\frac{\pi z}{2}))}{z^2-1}dz$ where c is the cycle $|z|=2$ in positive direction.

According the roots of denominator polynomial, we have

$$\frac{2}{z^2-1}=\frac{1}{z-1}-\frac{1}{z+1}$$

Thus

$$\oint_c \frac{2(1+\sin(\frac{\pi z}{2}))}{z^2-1}dz = \oint_c \frac{1+\sin(\frac{\pi z}{2})}{z-1}dz + \oint_c \frac{1+\sin(\frac{\pi z}{2})}{z+1}dz$$

$$=2\pi i(1+\sin(\frac{\pi}{2})) + 2\pi i(1+\sin(\frac{-\pi}{2})) = 4\pi i$$

Example 15: find $\oint_c \frac{2zcos\pi z}{z^3+4z^2-3z-18}dz$ where c is the cycle $|z|=5$ in positive direction.

We have

$$\frac{2z}{z^3+4z^2-3z-18} = \frac{\frac{4}{25}}{z-2} - \frac{\frac{4}{25}}{z+3} + \frac{\frac{6}{5}}{(z+3)^2}$$

Therefore

$$\oint_c \frac{2zcos\pi z}{z^3+4z^2-3z-18}dz = \oint_c \frac{\frac{4}{25}\cos\pi z}{z-2}dz - \oint_c \frac{\frac{4}{25}\cos\pi z}{z+3}dz + \oint_c \frac{\frac{6}{5}\cos\pi z}{(z+3)^2}dz$$

$$=2\pi i(\frac{4}{25}\cos 2\pi) - 2\pi i(\frac{4}{25}\cos(-3\pi)) + 2\pi i(\frac{-6\pi}{5}\sin(-3\pi)) = \frac{16}{25}\pi i$$

Example 16: find $\oint_c \frac{1-2e^z}{z^4-2z^3+z^2}dz$ where c is the cycle $|z|=2$ in positive direction.

We have

$$\frac{1}{z^4-2z^3+z^2} = \frac{2}{z} + \frac{1}{z^2} - \frac{2}{z-1} + \frac{1}{(z-1)^2}$$

Thus

$$\oint_c \frac{1-2e^z}{z^4-2z^3+z^2}dz = \oint_c \frac{2(1-2e^z)}{z}dz + \oint_c \frac{1-2e^z}{z^2}dz - \oint_c \frac{2(1-2e^z)}{z-1}dz + \oint_c \frac{1-2e^z}{(z-1)^2}dz$$

$$=2\pi i(2(1\text{-}2e^{\circ})) + 2\pi i(\text{-}2e^{\circ}) - 2\pi i(2(1-2e)) + 2\pi i(-2e) = 12\pi i(e-1)$$

Note that integration with partial-fraction decomposition and its generalization, according to Cauchy-Goursat theorem or Cauchy's integral formula, can be expanded and used for multiple connected domains. [2]

Example 17: evaluate $\oint_{\partial D} \frac{z^2}{z^4+2z^2+1}dz$ where ∂D is the contour of multiply- connected domain $\frac{1}{2} < |Z| < 2$ in positive direction.

To solve this, we have

$$\frac{z^2}{z^4+2z^2+1} = \frac{\frac{-i}{4}}{(z-i)} + \frac{\frac{1}{4}}{(z-i)^2} + \frac{\frac{i}{4}}{(z+i)} + \frac{\frac{1}{4}}{(z+i)^2}$$

Therefore,

$$\oint_{\partial D} \frac{z^2}{z^4+2z^2+1}dz = \oint_{\partial D} \frac{\frac{-i}{4}}{(z-i)}dz + \oint_{\partial D} \frac{\frac{1}{4}}{(z-i)^2}dz + \oint_{\partial D} \frac{\frac{i}{4}}{(z+i)}dz + \oint_{\partial D} \frac{\frac{1}{4}}{(z+i)^2}$$

$$=2\pi i(\frac{-i}{4}) + \circ + 2\pi i(\frac{i}{4}) + \circ = \circ$$

Example 18: find $\oint_{\partial D} \frac{\exp(z-2)}{(z^2-5z+6)^2}dz$ where ∂D is the contour of multiply- connected domain $1 < |z| < 4$ in positive direction.

We have

$$\frac{1}{(z^2-5z+6)^2} = \frac{2}{z-2} + \frac{1}{(z-2)^2} - \frac{2}{z-3} + \frac{1}{(z-3)^2}$$

$$\oint_{\partial D} \frac{\exp(z-2)}{(z^2-5z+6)^2}dz = \oint_{\partial D} \frac{2\exp(z-2)}{(z-2)}dz + \oint_{\partial D} \frac{\exp(z-2)}{(z-2)^2}dz - \oint_{\partial D} \frac{2\exp(z-2)}{(z-3)}dz + \oint_{\partial D} \frac{\exp(z-2)}{(z-3)^2}dz$$

$$=2\pi i(2\exp(\circ)) + 2\pi i(\exp(\circ)) - 2\pi i(2\exp(1)) + 2\pi i(\exp(1)) = 6\pi i(1\text{-}e)$$

Example 19: evaluate $\oint_{\partial D} \frac{Logz}{(z^3+iz^2+z+i)}dz$ where ∂D is the contour of multiply- connected domain $1 < |z-3| < 2$ in positive direction. (Let $L\log z$ be on the principal branch)

We have

$$\frac{1}{z^3+iz^2+z+i} = \frac{\frac{-1}{4}}{z-i} + \frac{\frac{1}{4}(3-i)}{(z+i)} + \frac{\frac{-1}{4}}{(z+i)^2}$$

Thus

$$\oint_{\partial D} \frac{L\log z}{(z^3+iz^2+z+i)}dz = \oint_{\partial D} \frac{\frac{-1}{4}L\log z}{z-i}dz + \oint_{\partial D} \frac{\frac{1}{4}(3-i)L\log z}{(z+i)}dz - \oint_{\partial D} \frac{\frac{1}{4}L\text{ogz}}{(z+i)^2}dz$$

$$=2\pi i(\frac{-1}{4}\log(i)) + 2\pi i(\frac{1}{4}(3-i)\log(-i)) - 2\pi i(\frac{1}{4}i) = \frac{\pi^2}{4}(4-i) + \frac{\pi}{2}$$

DISCUSSION

Real integration with partial-fraction decomposition can be applicable for rational functions, while complex integration $\oint_c f(z)dz$ can be used on simple closed contour c or multiply- connected domain D whether $f(z)$ be rational or in the form $\frac{g(z)}{p_n(z)}$ where $g(z)$ is analytic at each point inside and on simple closed contour c or multiply- connected domain D. Also calculating the integral by this method is much simpler and easier than using residue theorem or Laurent expansions methods. Our proposed method makes Cauchy or Cauchy-Goursat theorem or Cauchy's integral formula more efficient.

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